A Class of Integral Operators Generated by Random Variables

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Abstract: Let $\mathbf{X}$ be a random variable in $\mathbb{R}^p$ distributed symmetrically about zero with cumulants of order 4, 8, 12, ... equal to zero. This class of random variables includes the multivariate normal. Consider the linear integral operator $\mathbf{K}_{\mathbf{X}}$ defined by

$$\mathbf{K}_{\mathbf{X}} \mathbf{g}(\mathbf{x}) = \mathbb{E} [\mathbf{g}(\mathbf{x} + \mathbf{X})] = \int \mathbf{g}(\mathbf{x} + \mathbf{y}) dP(\mathbf{X} \leq \mathbf{y})$$

acting on the space of functions $\mathbf{g} : \mathbb{C}^p \rightarrow \mathbb{C}^q$ with Taylor series expansions about zero. By Fredholm theory, non-degenerate integral operators in L_2 generally do not have inverses. But $\mathbf{K}_{\mathbf{X}}$ is not in L_2 . We show that $\mathbf{K}_{\mathbf{X}}$ has inverse $\mathbf{K}_{\mathbf{X}}^{-1} = \mathbf{K}_{i\mathbf{X}}$, $i = \sqrt{-1}$.

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1 Introduction and summary

Let $\mathbf{X}$ be a random variable in $\mathbb{R}^p$. Consider the linear integral operator $\mathbf{K}_{\mathbf{X}}$ defined by $\mathbf{K}_{\mathbf{X}} \mathbf{g}(\mathbf{x}) = \mathbb{E} [\mathbf{g}(\mathbf{x} + \mathbf{X})]$ acting on the space of functions $\mathbf{g} : \mathbb{C}^p \rightarrow \mathbb{C}^q$. Does this operator have an inverse, and if so, what is it?

This is an important question to answer. Its answers could have wide spread applications in statistics and probability. In particular, with respect to characteristic functions and their inverses. Characteristic functions are widely used to manipulate distributions and their moments. Characteristic functions and their inverses are also used for fitting probability distributions to samples of data.

In Section 2, we give a class of random variables symmetric about zero for which $\mathbf{K}_{\mathbf{X}}^{-1} = \mathbf{K}_{i\mathbf{X}}$, $i = \sqrt{-1}$. This class includes the multivariate normal. Expansions for the density about the normal density are given in Section 3. This class is extended in Section 4 to symmetric random variables centered about some other mean. Section 5 considers Fredholm equations.

For most excellent accounts about linear operators and random variables, we refer the readers to Janson (1997), Taylor (2006) and Pietsch (2007).

2 Main result

Suppose $\mathbf{X}$ is a random variable in $\mathbb{R}^p$ with distribution F distributed symmetrically about zero with moment generating function $M_{\mathbf{X}}(\mathbf{t}) = \mathbb{E} [\exp(\mathbf{t}'\mathbf{X})]$ finite in some neighborhood of zero in $\mathbb{C}$. Let $\mathbb{N}$ denote the set of non-negative integers. The cumulants of $\mathbf{X}$, $\{\kappa_{\mathbf{n}\mathbf{X}}, \mathbf{n} \in \mathbb{N}^p\}$, are defined by its cumulant generating function

$$\sum_{\mathbf{n} \in \mathbb{N}^p} \kappa_{\mathbf{n}\mathbf{X}} \mathbf{t}^{\mathbf{n}} / \mathbf{n}! = \ln M_{\mathbf{X}}(\mathbf{t}) = K_{\mathbf{X}}(\mathbf{t})$$

say, where, for $\mathbf{n} \in \mathbb{N}^p$ and $\mathbf{t} \in \mathbb{C}^p$,

$$\mathbf{t}^{\mathbf{n}} = \prod_{k=1}^p t_k^{n_k}, \quad \mathbf{n}! = \prod_{k=1}^p n_k!, \quad |\mathbf{n}| = \sum_{k=1}^p n_k.$$

For functions $\mathbf{g} : \mathbb{C}^p \rightarrow \mathbb{C}^q$, consider the linear integral operator

$$\mathbf{K}_{\mathbf{X}} \mathbf{g}(\mathbf{x}) = \mathbb{E} [\mathbf{g}(\mathbf{x} + \mathbf{X})], \quad (2.1)$$

where the expectation is assumed to exist. If F has density f with respect to Lebesgue measure on $\mathbb{R}^p$, then

$$\mathbf{K}_{\mathbf{X}} \mathbf{g}(\mathbf{x}) = \int \mathbf{g}(\mathbf{t}) f(\mathbf{t} - \mathbf{x}) d\mathbf{x},$$

so that $\mathbf{K}_{\mathbf{X}}$ has kernel $k(\mathbf{x}, \mathbf{t}) = f(\mathbf{t} - \mathbf{x})$ with respect to Lebesgue measure on $\mathbb{R}^p$. But

$$\int \int k(\mathbf{x}, \mathbf{t})^2 d\mathbf{x} d\mathbf{t} = \infty,$$

so that Fredholm theory does not apply.

Theorem 2.1 *Suppose that $\mathbf{X}$ is a random variables in $\mathbb{R}^p$ symmetric about zero with finite moment generating function in some neighborhood of zero, and cumulants of order 4, 8, 12, ... equal to zero, that is,*

$$\kappa_{\mathbf{n}\mathbf{X}} = 0 \text{ for } |\mathbf{n}| \text{ divisible by } 4. \quad (2.2)$$

Consider the space of functions $\mathbf{g} : \mathbb{C}^p \rightarrow \mathbb{C}^q$, with Taylor series expansions converging in some neighborhood of zero, i.e. functions that are analytic in some neighborhood of zero. Then the linear integral operator $\mathbf{K}_{\mathbf{X}}$ of (2.1) acting on this space, has inverse operator $\mathbf{K}_{i\mathbf{X}}$, where $i = \sqrt{-1}$.

Proof: For $\mathbf{X}$ and $\mathbf{Y}$ two independent random variables in $\mathbb{R}^p$ with finite moments,

$$\mathbf{K}_{i\mathbf{Y}} \mathbf{K}_{\mathbf{X}} \mathbf{x}^{\mathbf{n}} = \mathbb{E} [(\mathbf{x} + \mathbf{X} + i\mathbf{Y})^{\mathbf{n}}] = \mathbf{x}^{\mathbf{n}}$$

if and only if $\mathbb{E} [\exp\{(\mathbf{x} + \mathbf{X} + i\mathbf{Y})'\mathbf{t}\}] = \exp(\mathbf{x}'\mathbf{t})$ in some neighborhood of zero. That is, $\mathbf{K}_{i\mathbf{Y}} = \mathbf{K}_{\mathbf{X}}^{-1}$ if and only if $M_{\mathbf{X}}(\mathbf{t})M_{i\mathbf{Y}}(\mathbf{t}) \equiv 1$ in some neighborhood of zero if and only if $\mathbf{K}_{\mathbf{X}}(\mathbf{t}) + \mathbf{K}_{i\mathbf{Y}}(\mathbf{t}) \equiv \mathbf{0}$ if and only if $\mathbf{Y}$ is symmetric about zero and for $|\mathbf{n}|$ even, $\kappa_{\mathbf{n}\mathbf{Y}} = (-1)^{|\mathbf{n}|/2} \kappa_{\mathbf{n}\mathbf{X}}$. But this holds by (2.2) if $\mathbf{Y}$ and $\mathbf{X}$ have the same distribution. This completes the proof. $\square$

Example 2.1 (*The multivariate normal.*) Suppose $\mathbf{X} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{V})$. This has density with respect to Lebesgue measure and moment generating function

$$\phi_{\mathbf{V}}(\mathbf{x}) = (2\pi)^{-p/2} (\det \mathbf{V})^{-1/2} \exp(-\mathbf{x}' \mathbf{V}^{-1} \mathbf{x} / 2), \quad M_{\mathbf{X}}(\mathbf{t}) = \exp(\mathbf{t}' \mathbf{V} \mathbf{t} / 2).$$

In this case,

$$\mathbf{K}_{i\mathbf{X}} \mathbf{x}^{\mathbf{n}} = \tilde{H}_{\mathbf{n}}(\mathbf{x}, \mathbf{V}) = H_{\mathbf{n}}(\mathbf{V}^{-1} \mathbf{x}, \mathbf{V}^{-1}), \quad H_{\mathbf{n}}(\mathbf{x}, \mathbf{V}) = \tilde{H}_{\mathbf{n}}(\mathbf{V}^{-1} \mathbf{x}, \mathbf{V}^{-1}),$$

where $H_{\mathbf{n}}(\mathbf{x}, \mathbf{V})$ and $\tilde{H}_{\mathbf{n}}(\mathbf{x}, \mathbf{V})$ are the multivariate Hermite and dual Hermite polynomials. See Withers (2000). Also

$$\mathbf{K}_{\mathbf{X}} \mathbf{x}^{\mathbf{n}} = \tilde{H}_{\mathbf{n}}^*(\mathbf{x}, \mathbf{V}) = H_{\mathbf{n}}^*(\mathbf{V}^{-1} \mathbf{x}, \mathbf{V}^{-1}),$$

where $H_{\mathbf{n}}^*(\mathbf{x}, \mathbf{V})$ and $\tilde{H}_{\mathbf{n}}^*(\mathbf{x}, \mathbf{V})$ are the modified multivariate Hermite and dual Hermite polynomials. That is, all negative signs are changed to positive signs. For example, with $p = 1$,

$$\begin{aligned} H_n(x, V) &= V^{-n} \tilde{H}_n(x, V), \quad H_n^*(x, V) = V^{-n} \tilde{H}_n^*(x, V), \\ \tilde{H}_2(x, V) &= x^2 - V, \quad \tilde{H}_2^*(x, V) = x^2 + V, \\ H_2(x, V) &= V^{-2} (x^2 - V), \quad H_2^*(x, V) = V^{-2} (x^2 + V). \end{aligned}$$

We note in passing a simple proof of the bi-orthogonality result due to Takemura and Takeuchi (1988):

Lemma 2.1 For $\mathbf{m}, \mathbf{n} \in \mathbb{N}^p$

$$\int \tilde{H}_{\mathbf{m}}(\mathbf{x}, \mathbf{V}) H_{\mathbf{n}}(\mathbf{x}, \mathbf{V}) \phi_{\mathbf{V}}(\mathbf{x}) d\mathbf{x} = \begin{cases} \mathbf{m}!, & \text{if } \mathbf{m} = \mathbf{n}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.3)$$

Proof: For $\mathbf{s}, \mathbf{t} \in \mathbb{C}^p$ set $\mathbf{a} = \mathbf{s} + \mathbf{V}^{-1} \mathbf{t}$. Then $\mathbf{a}' \mathbf{V} \mathbf{a} - \mathbf{s}' \mathbf{V} \mathbf{s} - \mathbf{t}' \mathbf{V}^{-1} \mathbf{t} = 2\mathbf{s}' \mathbf{t}$. So, for $\mathbf{Y} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{V}^{-1})$ independently of $\mathbf{X}$,

$$\begin{aligned} \int \mathbb{E} \left[\exp \left\{ (\mathbf{x} + i\mathbf{X})' \mathbf{s} + (\mathbf{V}^{-1} \mathbf{x} + i\mathbf{Y})' \mathbf{t} \right\} \right] \phi_{\mathbf{V}}(\mathbf{x}) d\mathbf{x} &= M_{\mathbf{X}}(i\mathbf{s}) M_{\mathbf{X}}(i\mathbf{V}^{-1} \mathbf{t}) M_{\mathbf{X}}(\mathbf{a}) \\ &= \exp(\mathbf{s}' \mathbf{t}). \end{aligned} \quad (2.4)$$

Taking the coefficient of $\mathbf{s}^{\mathbf{m}} \mathbf{t}^{\mathbf{n}}$ gives (2.3). $\square$

It would be interesting to obtain there an extension of (2.3) - or equivalently of (2.4) - to non-normal $\mathbf{X}$.

We now look at the class of random variables given by the theorem in more detail for the case $p = 1$. Their cumulant generating function is

$$K_X(t) = \sum_{n=0}^{\infty} \kappa_{4n+2} t^{4n+2} / (4n+2)!, \quad (2.5)$$

where $\kappa_n = \kappa_{nX}$. The cumulants κ_{4n+2} are not arbitrary as they are restricted by moment conditions like

$$\mathbb{E} \left[X^{k_0} \left(X^{k_1} - \mu_{k_1} \right)^2 \left(X^{k_2} - \mu_{k_2} \right)^2 \cdots \right] \geq 0$$

for $k_0, k_1, \dots$ even integers. If $\int |M_X(it)| dt < \infty$ then the distribution and density of X are given by

$$\begin{aligned} \frac{F(x) - F(x-h)}{h} &= \frac{1}{2\pi} \int (ith)^{-1} \{1 - \exp(ith)\} \exp\{-itx + K_X(it)\} dt, \\ f(x) &= \frac{1}{2\pi} \int \exp\{-itx + K_X(it)\} dt. \end{aligned} \quad (2.6)$$

See, for example, Theorem 6.2.3 of Chung (1974). One can now substitute (2.5) and integrate numerically for given cumulants - using, for example, Hermite integration - see equation (25.4.46) of Abramowitz and Stegun (1964) - or simply a package like MAPLE.

[Figures 2.1 and 2.2 about here.]

Example 2.2 *The simplest example is the one parameter family with variance one and sixth cumulant λ :*

$$K_X(t) = t^2/2 + \lambda t^6/6!.$$

Its first nine non-zero moments are

$$\begin{aligned} \mu_2 &= 1, \quad \mu_4 = 3, \quad \mu_6 = 3 \cdot 5 + \lambda, \quad \mu_8 = 3 \cdot 5 \cdot 7 + 4 \cdot 7\lambda, \\ \mu_{10} &= 3 \cdot 5 \cdot 7 \cdot 9 + 2 \cdot 5 \cdot 7 \cdot 9\lambda, \quad \mu_{12} = 3 \cdot 7 \cdot 11 (2\lambda^2 + 60\lambda + 45), \\ \mu_{14} &= 3 \cdot 7 \cdot 11 \cdot 13 (14\lambda^2 + 15 \cdot 7\lambda + 45), \quad \mu_{16} = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 15 (16 \cdot 7\lambda^2 + 168\lambda + 45), \\ \mu_{18} &= 3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 (56\lambda^3 + 2520\lambda^2 + 945\lambda + 675), \end{aligned}$$

as is well known for the normal (the case $\lambda = 0$). We now show that

$$\lambda > -0.27045 \dots \quad (2.7)$$

This follows from

$$0 < \mathbb{E} \left[(X^8 - \mu_8)^2 \right] = \mu_{16} - \mu_8^2 = 7 (a\lambda^2 + b\lambda + c)$$

for $a = 7 \cdot 16 \cdot 507$, $b = 7 \cdot 8 \cdot 15 \cdot 1286$, $c = 15^2 \cdot 1820$ since the lower root is $-18.753 \dots$. Weaker results are given by

$$0 < \mu_{2n} \text{ for } n = 3, 4, \dots, \quad 0 < \mathbb{E} \left[(X^{2n} - \mu_{2n})^2 \right] \text{ for } n = 2, 3.$$

Other inequalities are given by

$$\mathbb{E} [\mathbf{Y}\mathbf{Y}'] > \mathbf{0} \text{ for } \mathbf{Y}' = (X, X^2, \dots, X^r).$$

For example, with $r = 3$ this gives

$$0 < \mu_2\mu_6 - \mu_4^2 = 6 + \lambda.$$

However, the bound (2.7) is not sufficient since the density was found to be negative using MAPLE in (4.8, 5.6) for $\lambda = 1$, and in (6.3, 7.0) for $\lambda = 1/2$, and at 13 for $\lambda = 1/10$. Figure 2.1 plots the density for $\lambda = 0$, the normal and for $\lambda = 1$ using MAPLE. The densities appear almost identical. However, Figure 2.2 magnifies the portion in (4, 6) to show that the density turns negative for part of this range when $\lambda = 1$. However, MAPLE is not infallible: it gave the density as infinite for selected negative values of λ , in clear contradiction to (2.6).

3 Expansions about the normal

If the non-zero cumulants are small compared with the covariance $\mathbf{V}$, then one may expand about the normal: $\mathbf{X}$ has density

$$f_{\mathbf{X}}(\mathbf{x}) = Q(-\partial/\partial\mathbf{x}) \phi_{\mathbf{V}}(\mathbf{x})$$

for $Q(\mathbf{t}) = \exp\{K_{\mathbf{X}}(\mathbf{t}) - \mathbf{t}'\mathbf{V}\mathbf{t}/2\}$. Consider again:

Example 3.1 The case $p = 1$, $K_X(t) = t^2/2 + \nu t^6$ for $\nu = \kappa_6/6!$. So,

$$Q(t) = \exp(\nu t^6) = \sum_{k=0}^{\infty} \nu^k t^{6k}/k!,$$

$$f_X(x) = \phi(x) \sum_{k=0}^{\infty} \nu^k H_{6k}(x, 1)/k!$$

for $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

In the general case $p = 1$, we have the following.

Theorem 3.1 For $p = 1$,

$$Q(t) = \sum_{m=0}^{\infty} A_m t^{2m},$$

$$f_X(x) = \phi_V(x) \sum_{m=0}^{\infty} A_m H_{2m}(x, V),$$

where $V = \text{var}(X)$, $k_n/n! = \kappa_{4n+2}/(4n+2)!$, $\mathbf{k} = (k_1, k_2, \dots)$ and

$$A_m = \sum \{B_{n, m-2n}(\mathbf{k})/n! : 0 \leq n \leq m/2\},$$

where $B_{nj}(\mathbf{k})$ is the partial exponential Bell polynomial table on page 307 of Comtet (1974) as a function of $\mathbf{k}$. Here, $f_X(\cdot)$ is assumed to exist and the infinite sums are assumed to converge.

Proof: In the case $p = 1$,

$$K_X(t) = \sum_{n=0}^{\infty} t^{4n+2} k_n/n! = k_2 t^2 + S,$$

where

$$S = t^2 \sum_{n=1}^{\infty} t^{4n} k_n/n!.$$

So, $Q(t) = \exp(S)$. But by definition

$$S^j/j! = t^{2j} \sum_{n=j}^{\infty} B_{nj}(\mathbf{k}) t^{4n}/n!$$

for $j = 0, 1, \dots$. The result follows. $\square$

For an extension to general p , write $K_{\mathbf{X}}(\mathbf{t}) = \mathbf{t}'\mathbf{V}\mathbf{t}/2 + S$ for $S = \sum \{\kappa_{\mathbf{n}}\mathbf{t}^{\mathbf{n}}/\mathbf{n}!, |\mathbf{n}| \in 6 + 4\mathbb{N}\}$.

Example 3.2 Suppose that only cumulants of order two and six are non-zero. Then

$$\begin{aligned} S &= \sum \{\kappa_{\mathbf{n}}\mathbf{t}^{\mathbf{n}}/\mathbf{n}! : |\mathbf{n}| = 6\}, \\ S^j/j! &= \sum \{C_{\mathbf{n}}\mathbf{t}^{\mathbf{n}} : |\mathbf{n}| = 6j\} \text{ say }, \\ f_{\mathbf{X}}(\mathbf{x}) &= \phi_{\mathbf{V}}(\mathbf{x}) \sum_{j=0}^{\infty} \sum \{C_{\mathbf{n}}H_{\mathbf{n}}(\mathbf{x}, \mathbf{V}) : |\mathbf{n}| = 6j\} = \phi_{\mathbf{V}}(\mathbf{x}) \sum \{C_{\mathbf{n}}H_{\mathbf{n}}(\mathbf{x}, \mathbf{V}) : |\mathbf{n}| \in 6\mathbb{N}\}. \end{aligned}$$

4 An extension to general symmetric random variables

Is there an extension of the theorem to random variables that are not symmetric about zero? An extension to random variables that are symmetric about a constant say $\boldsymbol{\mu}$, is easy.

Example 4.1 Suppose that $\mathbf{X}$ is a constant. Then $\mathbf{K}_{\mathbf{X}}^{-1} = \mathbf{K}_{-\mathbf{X}}$.

For any independent random variables $\mathbf{X}, \mathbf{Y}$,

$$\mathbf{K}_{\mathbf{X}}\mathbf{K}_{\mathbf{Y}} = \mathbf{K}_{\mathbf{Y}}\mathbf{K}_{\mathbf{X}} = \mathbf{K}_{\mathbf{X}+\mathbf{Y}}$$

since

$$\mathbf{K}_{\mathbf{X}}\mathbf{K}_{\mathbf{Y}} \mathbf{g}(\mathbf{x}) = \mathbb{E} [\mathbf{g}(\mathbf{x} + \mathbf{X} + \mathbf{Y})].$$

So, if $\boldsymbol{\mu}$ is a constant in $\mathbb{C}^p$, and $\mathbf{X}_1, \dots, \mathbf{X}_m$ are independent random variables in $\mathbb{R}^p$ symmetric about zero and each satisfying (2.2), then

$$\mathbf{K}_{\boldsymbol{\mu}+\mathbf{S}_m}^{-1} = \begin{cases} \mathbf{K}_{-\boldsymbol{\mu}}\mathbf{K}_{i\mathbf{X}_1} \cdots \mathbf{K}_{i\mathbf{X}_m}, & \text{for } \mathbf{S}_m = \sum_{k=1}^m \mathbf{X}_k, \\ \mathbf{K}_{-\boldsymbol{\mu}}\mathbf{K}_{i\mathbf{X}}^m, & \text{if } \mathbf{X}_1, \dots, \mathbf{X}_m \text{ have the same distribution as } \mathbf{X}. \end{cases}$$

5 Fredholm equations

Consider functions $\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x}) : \mathbb{C}^p \rightarrow \mathbb{C}^q$ with Taylor expansions about zero. Let λ be a given constant. Suppose we wish to solve

$$(\mathbf{I} - \lambda\mathbf{K}_{\mathbf{X}})\mathbf{g} = \mathbf{h}$$

for $\mathbf{g}$ given $\mathbf{h}$. If $\mathbf{X}$ is symmetric about zero and satisfies (2.2), then this equation is equivalent to

$$(\mathbf{K}_{i\mathbf{X}} - \lambda\mathbf{I})\mathbf{g} = \mathbf{K}_{i\mathbf{X}}\mathbf{h}.$$

However, this gets us no closer to a solution. Fredholm theory does not apply since the kernel of $\mathbf{K}_{\mathbf{X}}$ is not in L_2 . A partial answer is the solution

$$\mathbf{g}(\mathbf{x}) = \left(\mathbf{I} + \sum_{k=0}^{\infty} \lambda^k \mathbf{K}_{\mathbf{X}}^k \right) \mathbf{h}(\mathbf{x}),$$

provided that the series converges.

It is an open question whether eigenvalues λ and eigenfunctions $\mathbf{g}(\mathbf{x})$ exist as solutions when $\mathbf{h}(\mathbf{x}) = \mathbf{0}$, and if so, to what extent to they satisfy the nice properties of Fredholm theory.

Clearly Fredholm's formula for the Fredholm determinant (giving the eigenvalues), fails, since even if the resolvent $K(\mathbf{x}, \mathbf{y}, \lambda) = f(\mathbf{x} - \mathbf{y}, \lambda)$ exists,

$$\int K(\mathbf{x}, \mathbf{x}, \lambda) d\mathbf{x} = \infty.$$

For more details on Fredholm theory in L_2 spaces, see, for example, Withers (1975).

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Figure 2.1 Densities for $\lambda = 0$ (the normal) and for $\lambda = -1$.

Figure 2.2 A closeup showing normal density and negative density for $\lambda = -1$.