Optimal Supplier Choice with Discounting

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Abstract

This paper investigates a model for pricing the demand for a set of goods when suppliers operate discount schedules based on total business value. We formulate the buyers's decision problem as a mixed binary integer program (MIP) which is a generalization of the capacitated facility location problem (CFLP). A branch and bound procedure using Lagrangean relaxation and subgradient optimization is developed for solving large-scale problems that can arise when suppliers' discount schedules contain multiple price breaks. Results of computer trials on specially adapted large benchmark instances of the CFLP, confirm that a subgradient optimization procedure based on Shor and Zhurbenko's r-algorithm, which employs a space dilation strategy in the direction of the difference between two successive subgradients, can solve such instances efficiently.

1. Introduction

The rapid development of business-to-business (B2B) electronic markets has triggered the need for efficient algorithms to allow an internet agent to source, i.e. to price in real time, an order for goods from multiple competing suppliers. The task is often complicated by two factors: 1) *fixed costs* which serve to reduce the number of suppliers actually used to fulfill an order, and 2) *discounts schedules* offered by suppliers to encourage purchase of greater quantities. Such discount schedules may involve either the cancellation of a fixed charges, e.g. to cover carriage costs, or increasing percentage reductions off list price over a sequence of pricebreaks.

Our study is motivated by the requirements of an online supplier of pharmaceuticals to high street chemists (retail pharmacists). The company acts as an internet broker in the sense that it carries no stock but, on receiving an enquiry (tentative order) for quantities of pharmaceutical products, it polls a set of wholesalers to determine which suppliers to use taking into account all applicable discounts. It then provides in real time a price quotation in answer to the enquiry based on cost, but including a profit mark-up. The enquiry is converted to a firm order if the total price of the basket of goods is judged to

be acceptable by the customer. Thus the broker’s task is to source the order at least cost. Note that the terms “enquiry” and “order” will be regarded as synonymous below.

We assume that the company has online access to a negotiated “static” price list for each supplier, and there is no aggregation of customers’ enquiries or economies achievable through “bulk buying”, although in certain business contexts this would be an interesting possibility to consider.

We formulate below a general model for the “*buyer’s decision problem*” (*BDP*) incorporating two common types of discount offered by suppliers: *Type A* : a cancellation or reduction in the fixed charge, and *Type B*: a percentage off the list price of each item. Discounts based on the value of an order have been termed a “*business volume*” discount (*BVD*) in contrast to a “*total quantity*” discount (*TQD*) when price breaks are defined by *number of units* purchased.(Goossens et al., 2004). Discounting may also take the form either of an “*all units*” policy modelled here, or an “*incremental*” policy. For a perspective on discounting theory and practice, see (Munson and Rosenblatt, 1998) where a field study on 39 firms is reported. An optimization model for vendor selection in the presence of price breaks was reported in (Chaudhry et al., 1993), though at that time there was no requirement for an online tool. The precise form of discounting employed will depend on the application, however the methods we develop for *BVD* can be easily generalized to other cases.

We show that the optimal allocation of a basket of goods to a set of suppliers in the market is an integer programming problem which resembles the capacitated facility location problem (*CFLP*). A supplier’s price schedule represents a set of *per unit costs* and *fixed setup costs*. An interesting feature of our model is that the cost functions implied by the discount structures described here are distinctive, being discontinuous and in general neither concave nor convex. We note the resemblance to “staircase cost functions” proposed for modelling production and distribution costs when a plant can be constructed in a range of sizes on a single site incurring scale-dependent setup (and running) costs (Holmberg, 1994), (Holmberg and Ling, 1997). Unlike our situation however, their fixed costs *increase* monotonically with plant capacity.

The fast response time required in an online context motivates the need for an efficient computational procedure for solving *BDP*. Whilst the efficient solution of large scale instances of *CFLP* has been the subject of much research in over many years (Cornuejols et al., 1991), (Agar and Salhi, 1998), (Bornstein and Azlan, 1998), the development of more efficient solution heuristics for large scale problems remains an area of active research, see for example (Barahona and Chudak, 2005) and (Klose and Goertz, 2007). Such research has focussed almost exclusively on Lagrangean relaxation techniques, see (Krarup

and Bilde, 1977), (Beasley, 1988) and (Körkel, 1989). In this paper we report the results of investigating several new heuristics for finding tight lower bounds for the Lagrangean dual problem for *BDP* involving subgradient optimization. A novel feature of our study is the use of the “*r-algorithm*” proposed by Shor and Zhurbenko (Shor, 1998), (Shor and Zhurbenko, 1971) which employs space dilation techniques to implement the subgradient optimization. We note the recent development of memoryless space dilation techniques in (Sherali et al., 2001) and related work by (Wu et al., 2006) in the context of *CFLP* with general setup costs

An outline of our paper follows. We first give some motivating examples to further illustrate the practical context of our study. We then formulate the *buyer’s decision problem (BDP)* as a mixed integer linear program assuming an *all units BVD* discount policy. We show that a transformation making use of “*pseudosuppliers*” results in a variant of the capacitated facility location problem (*CFLP*) which may be solved for large scale instances (involving many suppliers and multiple price breakpoints) by an efficient Lagrangean heuristic, Shor’s *r-algorithm*. Such a procedure which employs the geometric concept of space dilation in the direction of two successive subgradients, represents an alternative to the “classical” subgradient approach. We present the results of computer experiments on a number of generated instances (a) to compare solution efficiencies of the *r-algorithm* with a classical subgradient method, and (b) to examine the time requirements of solution procedures under different levels and types of discounting. We conclude with a discussion and suggestions for future research.

2. Motivating examples

Example 1

A mobile phone company announces that customers making over £30 of calls in a month will receive a refund of £5. In addition, customers making over £50 of calls in a month will qualify for a 20% discount from published tariffs. The discount schedule contains three piecewise linear segments created from two price breakpoints at £30 and £50, representing “*all units*” discounts of *Type A* and *Type B* respectively. The price $p(v)$ of calls made to a total value v is given by

$$p(v) = \begin{cases} v & \text{if } v < 30 \\ v - 5 & \text{if } 30 \leq v < 50 \\ 0.8v - 5 & \text{if } 50 \leq v \end{cases}$$

and the graph given by the bold line in *Figure 1* has the typical sawtooth form (Sadrian and Yoon, 1994). From this graph we also observe the “*more for less*” phenomenon (Goossens et al., 2004) that it can be cheaper to make more calls if the value of telephone business is just less than either breakpoint, in order to benefit from the next discount regime. By contrast the dotted graph in *Figure 1* represents the price-value relationship under an “*incremental*” discount scheme in which only calls made above the £50 breakpoint qualify for the reduced tariff.

Example 2

Prices of boxes of tulips and roses from two florists for are given in Table 1. Delivery charges are £10 for florist 1 and £5 for florist 2. Charges are waived on orders over £50 in value. Optimal transportation matrices for a) 7 boxes of tulips and 3 boxes of roses, and b) 7 boxes of tulips *only* are given in Table 2.

In the optimal solution to a) roses are purchased from both florists, showing that single sourcing is sub-optimal. In the optimal solution to b) it is optimal to exceed demand by ordering an additional single rose from the first florist - illustrating the “*more for less*” phenomenon.

3. The buyer’s decision problem

Consider a set of n products (items) indexed by $j \in J$ and suppose that each item j can be supplied by a common set of suppliers $s \in S$ at unit cost c_j^s from supplier s . We refer to $c^s = (c_1^s, \dots, c_n^s)$ as the *list price* for items supplied by supplier s . Assume that each supplier offers discounts based on the total value of the order placed with that supplier computed according to the supplier’s list price. The amount and nature of the discount is governed by a sequence of q price bands. If we suppose, for ease of notation, that q is the same for each supplier, then given a set of breakpoints $V_0^s < V_1^s < \dots < V_q^s$ the price bands for supplier s are consecutive intervals $I_1^s = [V_0^s, V_1^s)$, ..., $I_q^s = [V_{q-1}^s, V_q^s)$. We will assume without loss of generality that $V_0^s = 0$ and $V_q^s = \infty$. An order placed with supplier s for x_j^s units of item j ($j \in S$) forms a vector $x^s = (x_1^s, \dots, x_n^s)$ and has *total business value (TBV)* given by $V^s(x^s) = \sum_{j \in J} c_j^s x_j^s$.

The discounted price $\pi^s(x^s)$ that supplier s charges for the order is assumed to take the form

$$\begin{aligned} \pi^s(x^s) &= f_k^s + \rho_k^s \sum_{j \in J} c_j^s x_j^s & \text{if } V^s(x^s) \in I_k^s \\ &= f_k^s + \sum_{j \in J} c_{kj}^s x_j^s & \text{say.} \end{aligned}$$

for given constants $\{\rho_k^s\}, \{f_k^s\}$, $k \in K = \{1, \dots, q\}$. We usually expect that higher order values will attract greater levels of discount so that $1 = \rho_1^s \geq \rho_2^s \geq \dots$ and $f_1^s \geq f_2^s \geq \dots$. Note that $1 = \rho_1^s$ means that $c_j^s = c_{1j}^s$, i.e. the first price band $k = 1$ corresponds to list price.

The broker receives an order comprising a set of demands for D_j units of item j ($j \in J$) and seeks to reallocate $d = (D_1, \dots, D_n)$ amongst the suppliers at minimum cost. The set of (sub)orders $\{x^s\}$ should meet the total demand

$$\sum_{s \in S} x^s \geq d \quad (1)$$

minimize the total cost of supply

$$C(x) = \sum_{s \in S} \pi^s(x^s) \quad (2)$$

by taking advantage of all suppliers' discounts. Minimizing (2) will of course tend to restrict the number of suppliers. We refer to this cost optimization as the buyer's decision problem (BDP) and note that x^s may be real or integer, depending on application context.

Let x_{kj}^s denote the quantity of item j ordered from supplier s in price band k . Now $V^s(x^s) = \sum_{j \in J} c_j^s x_j^s \in I_k^s$ for some $k = k_s$, say, and clearly $x_{kj}^s = 0$ whenever $k \neq k_s$, since it will be optimal only for one price band ever to be used for any supplier. Define

$$y_k^s = \begin{cases} 1 & \text{if } V^s(x^s) \in I_k^s \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

and re-index

$$y_1^1, \dots, y_q^1; y_1^2, \dots, y_q^2; \dots; y_1^s, \dots, y_q^s$$

as a single sequence $\{y_i\}_{i \in I}$. We henceforth refer to $I = \{1, \dots, m\}$ as a set of *pseudosuppliers*. A corresponding re-indexing of $f_k^s = f_i$, $c_{kj}^s = c_{ij}$, $x_{kj}^s = x_{ij}$ and $I_k^s = [L_i, U_i]$ leads to the following generalized capacitated facility location problem (CFLP) in which the capacity constraints represent lower and upper bounds on TBV .

$$\text{BDP:} \quad \min_{y_i, x_{ij}} \sum_{i \in I} \left\{ f_i y_i + \sum_{j \in J} c_{ij} x_{ij} \right\} \quad (4)$$

$$\text{s.t.} \quad L_i y_i \leq \sum_{j \in J} c_{ij}^0 x_{ij} \leq U_i y_i \quad \forall i \in I, \quad (5)$$

$$\sum_{i \in P^s} y_i \leq 1 \quad \forall s \in S, \quad (6)$$

$$\sum_{i \in I} x_{ij} \geq D_j \quad \forall j \in J, \quad (7)$$

$$y_i \in \{0, 1\} \quad \forall i \in I \quad (8)$$

$$x_{ij} \in \mathbb{Z}^+ \quad \forall i \in I, j \in J \quad (9)$$

where

P^s is the set of pseudosuppliers i corresponding to real supplier s , and

$c_{ij}^0 = c_{1j}^s$ is the unit list price of item j for pseudosupplier i for $i \in P^s$.

The objective function (4) is precisely that of a standard (either simple or capacitated) plant location model. (5) states that the TBV of the suborder supplied from pseudosupplier i should fall within the appropriate price band. Note that in contrast to the usual CFLP model (5) contains both lower and upper bounds. Constraint (6) ensures that at most one price list per supplier can appear in an optimal solution. The demand constraints (7) may be expressed either as equalities or as lower bounds. Our use of inequalities corresponds to the “more-for-less” formulation mentioned by Goossens et.al., (Goossens et al., 2004) Section 4.2) allowing for the possibility that it may be cheaper to over-fulfill demand in order to benefit from a higher level of discount.

In practice there may be stock limits that restricted availability of some product j from some supplier s . There may also be commercial reasons why a supplier s may wish to limit or deny availability of some product j within a price band k . We therefore include in our formulation of BDP the upper bound constraints

$$0 \leq x_{ij} \leq S_{ij}, \quad i \in P^s \quad (10)$$

for some constants $\{S_{ij}\}$. Finally we note that while x_{ij} are integer variables in this formulation of BDP, in other applications x_{ij} may be continuous, so that (9) will be replaced by $x_{ij} \in \mathbb{R}^+$.

4. Lagrangean duality

Since no optimal solution can employ more than one price band from a single supplier, we can technically remove the upper bounds in (5) and in fact all the constraints (6) from the formulation of BDP and any optimal solution to BDP will remain optimal under the new formulation. However retaining the additional constraints allows tighter dual bounds in the relaxations we describe below.

We write the constraints (7) in the form $D_j - \sum_{i \in I} x_{ij} \leq 0$ and introduce as a vector of corresponding Lagrange multipliers, the dual variables $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_j \geq 0, j \in J$. The Lagrangean dual problem (LDP) corresponding to BDP can be stated as

$$\text{LDP:} \quad \max_{\lambda_j \geq 0} F(\lambda) \quad (11)$$

where

$$F(\lambda) = \min_{y_i, x_{ij}} \left\{ \sum_{i=1}^m f_i y_i + \sum_{i=1}^m \sum_{j=1}^n (c_{ij} - \lambda_j) x_{ij} + \sum_{j=1}^n D_j \lambda_j \right\} \quad (12)$$

subject to (5),(6),(8),(9), and (10). For each $\lambda \geq 0$ this Lagrangean dual subproblem is a relaxation of BDP, and as is well-known, $F(\lambda)$ is a concave piecewise-linear function. This allows us to employ subgradient techniques to seek a constrained maximum of (11) which provides the best lower bound for problem BDP.

5. The Lagrangean heuristic

At any node of the branch and bound tree we will have partitioned I into the index sets K_0, K_1, K_2 such that

$$y_i = 0, \quad i \in K_0, \quad y_i = 1, \quad i \in K_1, \quad y_i \in \{0, 1\}, \quad i \in K_2 \quad (13)$$

Thus K_0, K_1 are the pseudosuppliers fixed closed, open respectively; K_2 are the undetermined pseudosuppliers. Letting $P_L = |K_1|$ and $P_U = |K_1 \cup K_2|$ we may add the explicit bounds

$$P_L \leq \sum_{i=1}^m y_i \leq P_U \quad (14)$$

on the total number of actual suppliers used.

The solution to the Lagrangean dual subproblem (12)-(14) for a prescribed vector of Lagrange multipliers λ can be reduced to solving two knapsack prob-

lems. The first knapsack problem is defined for each non-closed pseudosupplier $i \in K_1 \cup K_2$ and determines the contribution to the dual function (12) from pseudosupplier i if open :

$$\alpha_i(\lambda) = f_i + \min_{\{x_{ij}\}} \sum_{j=1}^n (c_{ij} - \lambda_j) x_{ij} \quad (15)$$

subject to (5) and (10) where $x_{ij} \in \mathbb{Z}^+$. We note that solving the 1-D integer knapsack is NP-hard (Martello and Toth, 1990). However, since our aim in solving the dual subproblem is to find a lower bound to BDP we may solve instead the continuous knapsack relaxation. The minimum of (15) subject to (5) and (10) over $x_{ij} \in \mathbb{R}^+$ is achieved by a greedy heuristic, in which we order the x_{ij} by non-decreasing value of the ratio $(c_{ij} - \lambda_j) / c_{ij}^0$ and set the components of x_{ij}^* in turn to their maximum value subject to (5) and (10). In case of infeasibility, we set $y_i = 0$ and $x_{ij} = 0$ for each j . The speed and simplicity of this solution to the continuous relaxation is of course at the expense of an increased duality gap for $x_{ij} \in \mathbb{R}$ compared to that for $x_{ij} \in \mathbb{Z}$.

The second knapsack problem is a minimization problem on the set of Boolean variables $\{y_i\}$

$$F(\lambda) = \min_{\{y_i\}} \sum_{i \in K_2} \alpha_i(\lambda) y_i + \sum_{i \in K_1} \alpha_i(\lambda) + \sum_{j=1}^n D_j \lambda_j \quad (16)$$

subject to (6) and (8). For each supplier s , let $\beta_s = \min_{i \in P^s \cap K_2} \{\alpha_i(\lambda)\}$ and form the corresponding list of pseudosuppliers $i_1, i_2, \dots$ in non-decreasing order of β_s . Define the sequence of partial sums $\{S_t\}$ by

$$\begin{aligned} S_0 &= \sum_{i \in K_1} \alpha_i(\lambda) + \sum_{j=1}^n D_j \lambda_j \\ S_1 &= S_0 + \alpha_{i_1}(\lambda) \\ &\vdots \\ S_t &= S_{t-1} + \alpha_{i_t}(\lambda) \end{aligned} \quad (17)$$

The smallest value of S_{t^*} such that $P_L \leq t^* \leq P_U$ provides an optimal solution y^* to (16) and hence the solution to (12) subject to the given constraints. Let Z_U be the value of the *incumbent* i.e. of the best feasible solution found so far. We decide the branch is fathomed if $S_{t^*} \geq Z_U$, otherwise we continue to develop this node.

Reduction tests

This straightforward procedure for solving the dual subproblem by solving two knapsack problems provides the opportunity to check whether each y_i variable can be fixed at 0 or 1 in subsequent branchings. Beasley develops Lagrangean heuristics for location problems in (Beasley, 1993) making extensive use of such rules first proposed in (Akinc and Khumawala, 1977). We have developed the following rules for the BDP:

Let M^* denote the set of pseudosuppliers that have $y_i^* = 1$ in the solution to the Lagrangean subproblem (12) - (14).

Close penalties

For $i \in K_2 \cap M^*$ so $y_i^* = 1$, consider the change in $F(\lambda)$, say $\Delta F(\lambda)|_{y_i=0}$, as a result of setting $y_i = 0$ (*closing* i). Let $\alpha_{\min}$ be the smallest value of $\alpha_i(\lambda)$ with $y_i^* = 0$ such that pseudosupplier l can be feasibly opened in the solution to (12), keeping all other variables unchanged. Forcing pseudosupplier i closed allows l to be opened if S_{t^*} is reduced as a result. If closing i causes the lower limit on cardinality of M^* to be violated, l is also forced to enter M^* . As a result of these considerations the increase in $F(\lambda)$ due to closing i is

$$\Delta F(\lambda)|_{y_i=0} = \begin{cases} -\alpha_i(\lambda) + \alpha_{\min}, & \text{if } |M^*| = P_L \\ -\alpha_i(\lambda) + \min\{0, \alpha_{\min}\}, & \text{if } |M^*| > P_L \end{cases} \quad (18)$$

and if $F(\lambda) + \Delta F(\lambda)|_{y_i=0} \geq Z_U$ we discard the subproblem (12)-(14) with $y_i = 0$ and fix $y_i = 1$ in all subsequent completions of this branch, i.e. we transfer i from K_2 to K_1 .

Open penalties

For any pseudosupplier $i \in K_2$ satisfying $y_i^* = 0$, we calculate the change $\Delta F(\lambda)|_{y_i=1}$ in $F(\lambda)$ as a result of forcing $y_i = 1$ (*opening* i). We need to distinguish between three cases. In the first case $i \in P_s$ and supplier s is already represented in M^* by some l with $y_l^* = 1$ (i.e. P_s is already open in y^* at some other level of discount). This case results in a forced exchange of i with l . In the second case, opening pseudosupplier i violates the upper limit on cardinality of M^* so that some open pseudosupplier $l = i_2$, say, must be closed. In the final case, opening i may allow the *possibility* that some supplier i_2 be closed. Let $\alpha_{\max}$ be the maximum of $\alpha_l(\lambda)$ over $l \in M^* \cap K_2$. In the latter two cases, the greatest reduction in $F(\lambda)$ will be achieved when i_2 is the corresponding argument to $\alpha_{\max}$. The net change in $F(\lambda)$ can be summarized

as follows:

$$\Delta F(\lambda)|_{y_i=1} = \begin{cases} \alpha_i(\lambda) - \alpha_l(\lambda), & \text{if } P_s \text{ is already open at } l \\ \alpha_i(\lambda) - \alpha_{\max}, & \text{if } |M^*| = P_U \\ \alpha_i(\lambda) - \max\{0, \alpha_{\max}\}, & \text{otherwise} \end{cases} \quad (19)$$

Narrowing the bounds P_L, P_U

The sequence $\{S_t\}$ given by (17) has the property that S_t is the value of the optimal dual solution $F(\lambda)$ subject to the cardinality constraint $|M^*| = t$. In other words, each such value denoted by $F_t(\lambda)$ is a lower bound for the original BDP with $\sum_{i=1}^m y_i = t$. For very little extra computation when solving the dual subproblem, it is possible to determine the maximal interval $[t_1, t_2]$ for which $F(\lambda) < Z_U \forall t \in [t_1, t_2]$. This allows us to reduce the interval $[P_L, P_U]$ to $[t_1, t_2]$. We note that $[t_1, t_2]$ will be non-empty since $S_{t^*} < Z_U$, otherwise this branch would have been fathomed on solving the dual subproblem.

Subgradient procedure

The family of "r-algorithms" (Shor and Stetsyuk, 2002) has been developed for the unconstrained maximization of concave objective functions over a continuous domain. The r-algorithm is a refinement of the classical subgradient method with space dilation developed by Shor and co-workers in the early 1970's (Shor, 1970) which Todd has identified as an example of a rank-one quasi-Newton method (Todd, 1986). The refinement employs space dilation in the direction of the difference between two successive subgradients. At the optimal solution $\{x_{ij}^*\}$ to the Lagrangean dual subproblem at iteration k with $\lambda = \lambda^k$, the value of the supergradient is

$$\nabla_j F(\lambda^k) = D_j - \sum_{i=1}^m x_{ij}^*, \quad j = 1, \dots, n$$

The constraints $\lambda_j \geq 0$ can be incorporated by the use of a symmetrizing transformation $\lambda_j = |u_j|$ where $u_j \in \mathbb{R}$. The computational efficiency of the r-algorithm depends on a choice of the space dilation coefficient α and tuning the values of the step multiplier. We have applied the r-algorithm with constant coefficient α of space dilation in the interval $[2, 4]$ and adaptive tuning of the step multiplier (see (Shor, 1998), p.104). This choice seeks to achieve as large as possible an improvement of the objective function along the current direction of search. In computer experiments reported below, timings are compared for the classical subgradient method and the r-algorithm on simulated instances. In our

experiments, no speed improvements were introduced by the recently developed memoryless space dilation and reduction strategy of (Sherali et al., 2001).

Generating feasible solutions

The branch-and-bound algorithm requires a heuristic to generate good feasible solutions in order to prune the tree by discarding manifestly sub-optimal sub-problems. Given a set of open pseudosuppliers M^* from the solution to the Lagrangean dual problem (11) we seek

$$Z(M^*) = \sum_{i \in M^*} f_i + \min_{x_{ij}} \sum_{i \in M^*} \sum_{j \in J} c_{ij} x_{ij} \quad (20)$$

subject to

$$L_i \leq \sum_{j \in J} c_{ij}^0 x_{ij} \leq U_i, \quad i \in M^* \quad (21)$$

$$\sum_{i \in M^*} x_{ij} \geq D_j, \quad j \in J \quad (22)$$

and constraints 9 and 10. This problem, whether x_{ij} is integer or real, is a generalized transportation problem with upper and lower bounds on the total value sourced from each pseudosupplier. In computer experiments, we have taken the above model with only the lower bound in (21) and used CPLEX to solve the continuous problem. The optimal solution is rounded up to the nearest integer solution. We then check whether the upper bound in (21) holds. If not, owing to the nature of our discount function we are able to find another pseudosupplier for which constraint (21) will be valid and for which the optimal value will be less.

Branching procedure

We have implemented two heuristics for choosing the branching variable from amongst the undecided variables $\{y_i\}_{i \in K_2}$. In the first approach we have used a heuristic procedure due to (Belyaeva et al., 1978) and not widely known, for estimating the optimal values of $\{y_i\}$ in the solution to the LP primal corresponding to the relaxed Lagrangean dual subproblem (12). The calculation is based on an estimate

$$y_i^* = \lim_{k \rightarrow \infty} \left\{ \frac{\sum_{t=1}^k h_t y_i^t}{\sum_{t=1}^k h_t} \right\} \quad (23)$$

using the values of y_i encountered during the subgradient iteration $\{\lambda^t\}_{t=0}^\infty$ which converges in the point λ^* . Here h_t and y_i^t are respectively the step-

length and value of y_i at step t of the iteration. If all y_i^* are integer, then no branching is needed. Otherwise we select the "most non-integral" y_i using the smallest value of $|y_i^* - \frac{1}{2}|$ as the criterion and branch first on the subproblem $y_i = 1$.

In our second approach which is new, we determined the branching variable y_i through the "Open" and "Close" penalties obtained after solving the dual subproblem. We define

$$\begin{aligned} F_i^O &= \max_k \{ F(\lambda^k) + \Delta F(\lambda^k) |_{y_i=1} \} \\ F_i^C &= \max_k \{ F(\lambda^k) + \Delta F(\lambda^k) |_{y_i=0} \} \end{aligned} \quad (24)$$

and branch by the variable i for which $F_i^C - F_i^O$ is maximal. If $F_i^C \geq F_i^O$, then we set $y_i = 1$ (open). Otherwise we set $y_i = 0$ (closed). Our computational experiments on Set 3 instances below have shown that such an assignment of y_i leads very fast to a feasible solution which is close to an optimal solution.

6. Computational experiments

We carried out computer experiments to compare the performance of the branch and bound procedure described above using two subgradient methods for finding lower bounds, the "r-algorithm" of Shor and the "classical" subgradient approach of Christofides and Beasley (Christofides and Beasley, 1983). In this comparison, three datasets were used. Problem sets 1 and 2 were adapted from three *uncapacitated facility location (UFL)* problem instances derived from the OR benchmarks library (Beasley, 1990): *Cap 71*, *Cap 101* and *Cap 131*. Problem set 3 was adapted from the instance *Uniform-123* in the Library of Discrete Location Problems maintained by the Sobolev Institute of Mathematics in Novosibirsk (Kochetov and Ivanenko, 2003). It is known to be a hard instance.

OR Lib - Set 1

This problem set is adapted from three instances in the OR benchmarks library: *Cap 71* with $m = 16$ plants, $n = 50$ customers; *Cap 101* with $m = 25$, $n = 50$; *Cap 131* with $m = n = 50$. A single breakpoint at $V = 50,000$ was introduced in all suppliers, when a fixed *Type A* discount of value 7500 became applicable. The range of variation of costs (unit prices) $\{c_{ij}\}$ is $[0, 1.4 \times 10^6]$ that of demands $\{D_j\}$ is $[31, 1.3 \times 10^4]$. We wished to compare the efficiency of the subgradient procedures and the sensitivity of each procedure to scaling of the demands $\{D_j\}$.

Table 3 summarizes the comparison. Detailed are the number of branchings, the total number of iterations and computing times for both methods (seconds on a PC with installed Intel Pentium 200 chip). Each instance was solved with and without scaling the demands $\{D_j\}$ to unity.

We see that the number of branchings is uniformly less for the r-algorithm, indicating that the r-algorithm returns tighter lower bounds than the classical subgradient method. Solution times however are not markedly different due to greater overheads in computing subgradients for the r-algorithm. Scaling before solution is effective for the simple subgradient method, but does not affect the performance of the r-algorithm.

OR Lib - Set 2

Table 4 presents CPU timings (s.) to solve instances from problem set 2. Each instance is derived from the OR Library dataset *CAP131* with $m = n = 50$ by adding different types and levels of discount to create two price bands. The level of discount offered in the second price band is indicated in the dataset name as a *percentage reduction*, respectively in fixed cost $\{f_i\}$ in the case of Type A and from the unit prices $\{c_{ij}\}$ in the case of Type B.

The results indicate that the r-algorithm generally takes more time to compute the bound than the classical subgradient method. The case A0+B0 is the undiscounted case which corresponds to *SPLP* without pseudosuppliers and is clearly very fast. Problems with Type A discounts require much more computing time to solve than instances with a purely Type B discount although the instance A100 which corresponds to the total removal of fixed costs for orders above the value threshold is also solved in less time. The most difficult cases are Type A instances with 50-60% discount. Timings are much reduced when both types of discount are applicable. This might be explained, intuitively, by the observation that when suppliers offers both types of discount simultaneously there is an incentive to use fewer suppliers, making the corresponding instance easier to solve.

Sobolev Institute - Set 3

For problem set 3, instances with $m = n = 100$ and different types/levels of discount were generated from a single dataset *Uniform-123* containing $m = 100$ plants and $n = 100$ customers with $D_j = 1$, each $j \in \{1, \dots, n\}$. Each supplier was given a single price break with $f_1^s = 3000$ and $f_2^s = 0$, giving rise to 200 pseudosuppliers. D_j was either left unscaled or generated as uniformly distributed pseudorandom numbers in the range $[1, D]$ where D is tabulated.

Table 5 shows CPU times (s.) taken to find *exact* and *approximate* solutions to within a prescribed level of accuracy. The value of the best feasible solution found so far, Z_U , is also given with * indicating termination occurs at a true minimum.

We observe that the classical subgradient approach solves many of our instances more efficiently than the r-algorithm. However, solution times are generally of the same order of magnitude, although in one instance (B50;3%) the r-algorithm took more than 50 times longer. We note, however that the performance of the r-algorithm is also influenced by different choices of tuning parameters. When scaling of $\{D_j\}$ is performed so the demand for different products varies, we see an increase in the solution times using the simple subgradient algorithm. However the solution times by the r-algorithm remain much more stable. The largest solution times for approximate solutions are observed on instances with the pure Type A discounts of 50%.

7. Conclusion

We have addressed a general problem of importance in e-commerce, how to determine a minimum cost assignment of an order for a basket of goods to a set of suppliers taking into account fixed charges and some common types of discounting policies. The fast response time required in an online context motivates the need for an efficient computational procedure.

Our integer programming formulation may be regarded as a new type of capacitated facility location model for which solution procedures based on Lagrangean relaxation have been extensively studied. Efficient procedures should be capable of rapidly solving large instances with possibly scores of suppliers offering hundreds of products and operating discount schedules containing multiple breakpoints. Two branch-and-bound algorithms have been implemented which differ in the method of computing the lower bounds. The first employs the dual-based Lagrangean heuristic based on the “classical” subgradient method of Bilde and Krarup (1977) and the second is the “r-algorithm” based on space dilation in the direction of difference of two consecutive supergradients due to Shor (1970). Details of a new branching rule and fathoming heuristics have been provided on the basis of Open and Close penalties akin to those first proposed by Khumawala (Khumawala, 1972), (Akinc and Khumawala, 1977).

Computational experiments show that

- although the r-algorithm produces tighter bounds giving a reduced branch and bound tree, the classical subgradient algorithm achieved comparable solution times on test problems.

- the “self-tuning” nature of the r-algorithm means however that solution times may be less sensitive to large differences in scale of the problem coefficients if this method is adopted.
- discounts of Type A (change in fixed costs) may present more of a computational challenge than Type B (% change in variable cost).

We conclude that Lagrangean relaxation techniques can efficiently solve to optimality large-scale instances of the buyer’s decision problem involving many hundreds of suppliers and price lists containing multiple price breakpoints. On the basis of the instances solved, it is unclear whether the extra programming required to implement the r-algorithm will be justified by faster solution speeds.

The methods developed here are the basis for further studies into online sourcing of goods with different discounting policies. The issue of *bulking* or aggregating orders is one we have not addressed here. In practice it is common for suppliers to operate more complex discounting policies, e.g. based on the total value of orders aggregated over a time window. Such an environment requires an optimal strategy that evolves over time. A supplier faced with current competition in a specific market for products and services may make use of these optimal solutions as a tool for evaluating alternative price lists and discounting strategies. We foresee natural developments of our model motivated by recent research into the theory and practice of reverse auctions.

Finally, we note that optimization models involving linear and fixed transaction costs have recently been proposed for portfolio optimization in the financial context. Such models incorporate a stochastic dimension as the objective function coefficients are rates of return which are assumed to occupy a probability space (Lobo et al., 2007).

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Figure 1:
BVD "sawtooth discount function"

Figure 1: BVD "sawtooth" discount function

Table 1:
Florists' price lists

	Delivery	Items		Discount
	charge	tulip	rose	breakpoint
Florist 1	10	7	6	50
Florist 2	5	9	3	50

Table 2
Optimal transportation solutions

(a) Solution to order for 7 roses & 3 tulips

	Delivery	tulip	rose	Value
Florist 1	0	7	1	55
Florist 2	5	0	2	6
Charge	5	49	12	<u>66</u>

(b) Solution to order for 7 roses

	Delivery	tulip	rose	Value
Florist 1	0	7	1	55
Florist 2	0	0	0	0
Charge	0	49	6	<u>55</u>

Table 3:

Comparison of subgradient procedures with and without scaling

ORLib instance	Classical subgradient						r-algorithm		
	no. suppliers	no. items	Scaling y/n	no. branchings	no. iterations	time	no. branchings	no. iterations	time
Cap 71	16	50	y	4	94	3	2	211	6
			n	21	1240	12	2	166	5
Cap101	25	50	y	6	43	4	2	201	7
			n	9	442	8	2	161	7
Cap131	50	50	y	9	115	5	4	250	15
			n	31	886	27	2	201	13

Table 4:

Effect of discount type and level on execution times (s.)

Cap131 derived instance	Timings (s.)			
	Classical		r-algorithm	
	Unscaled	Scaled	Unscaled	Scaled
A0+B0	1.8	1.4	2	3
A25	191	62	133	147
A40	548	325	545	666
A50	1413	755	2492	2167
A60	1702	958	2231	2246
A75	1082	316	1037	645
A100	275	36	110	151
B10	3	3	4	3
B15	6	2	4	3
B20	13	5	4	3
A25+B10	126	36	90	87
A50+B10	106	44	43	46
A75+B10	71	26	15	20
A100+B10	50	6	7	7

Table 5:

Exact and approximate solution times for "Uniform-123" instances

Uniform-123 + discount type	%	Range factor D	Classic subgradient		r-algorithm	
			Time s.	Value Z_U	Time s.	Value Z_U
A0+B0	exact		957	71342*	1379	71342*
$\vdots$	$\vdots$	2	2137	$\vdots$	1557	$\vdots$
$\vdots$	$\vdots$	1000	7065	$\vdots$	1612	$\vdots$
A0+B0	1%		383	72034	565	71342*
$\vdots$	2%		147	72606	314	72080
$\vdots$	3%		61	73377	78	73422
$\vdots$	$\vdots$	10	938	71342	403	72681
A30	3%		293	71716	1596	71344
A50	3%		903	68815	2836	68902
$\vdots$	5%		292	70396	411	69476
B20	3%		92	72282	154	71704
B50	$\vdots$		99	71086	5442	72375