Remarks on the Stable $S_\alpha(\beta, \gamma, \mu)$ Distribution

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Abstract. Explicit closed forms are derived for the probability density function of the stable distribution $S_\alpha(\beta, \gamma, \mu)$, $\alpha \in (1, 2]$. Consequent asymptotic expansions are given. The expressions involve the Srivastava-Daoust generalized Kampé de Fériet hypergeometric $\mathcal{S}$ -function, the Fox-Wright generalized hypergeometric Ψ -function, and the Gauss hypergeometric function ${}_2F_1$.

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1. Introduction and preliminaries

A random variable (r.v.) ξ is said to have the stable distribution if its characteristic function (CHF) is specified by [10, p. 8, Eq. (1.6)]:

$$\phi(t) = Ee^{it\xi} = \begin{cases} \exp \{ i\mu t - \gamma^\alpha |t|^\alpha [1 - i\beta \tan(\pi \frac{\alpha}{2}) \operatorname{sgn}(t)] \}, & \alpha \neq 1, \\ \exp \{ i\mu t - \gamma |t| (1 + i\beta \frac{2}{\pi} \ln t) \}, & \alpha = 1, \end{cases} \quad (1.1)$$

where $\alpha \in (0, 2]$, $|\beta| \leq 1$, $\gamma > 0$ and $\mu \in \mathbb{R}$. We write $\xi \sim S_\alpha(\beta, \gamma, \mu)$. The stable distribution class contains the Gaussian distribution ($\alpha = 2$), the Cauchy distribution ($\alpha = 1$, $\beta = 0$) and the Lévy distribution ($\alpha = 1/2$, $\beta = 1$) as particular cases. The Landau distribution ($\alpha = 1$, $\beta = 1$) and the dirac delta distribution ($\alpha \downarrow 0$, $\gamma \downarrow 0$) are also contained as particular cases.

Stable distributions have wide-ranging applications. Some of the application areas include the lighthouse problem, distribution of masses in space, random walks, hitting times for Brownian motion, fractional diffusions, stock prices, foreign exchange rates, value at risk, multiple assets, time series, signal processing, copulas, exponential power distributions, queueing theory, geology, physics, survival analysis and reliability, network traffic, computer science, biology and medicine, discrepancies, punctuated change, central pre-limit theorems, extreme values models, behavior of the sample mean and variance, and the appropriateness of infinite variance models. For discussion on these application areas and references, we refer the readers to [10, Chapter 2].

Because of its wide applicability, it is important that the properties of $\xi \sim S_\alpha(\beta, \gamma, \mu)$ are known explicitly. However, even the probability density function (PDF) of ξ has not been known explicitly except for the three particular cases of the Gaussian

distribution ($\alpha = 2$), the Cauchy distribution ($\alpha = 1, \beta = 0$) and the Lévy distribution ($\alpha = 1/2, \beta = 1$). There has been much development of approaches to compute the PDF numerically. We refer the readers to [19, 11, 3, 8, 9, 2, 12, 13, 14, 7, 6]. See also Lukács [5], Zolotarev [20], Uchaikin and Zolotarev [18] and [10, Chapter 3].

The aim of this note is to derive an explicit closed form for the PDF of $\xi \sim S_\alpha(\beta, \gamma, \mu)$ given by

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi(t) dt,$$

where $\phi(t)$ is given by (1.1). According to [10, p. 11, Theorem 1.9], stable distributions are continuous and infinitely differentiable. So, the PDF is well defined.

We shall need the following mathematical tools. Let ${}_p\Psi_q$ denote the Fox-Wright generalization of the hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters [17, p. 4, Eq. (2.4)] defined by

$${}_p\Psi_q \left[\begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| x \right] = \sum_{m=0}^{\infty} \frac{\prod_{\ell=1}^p \Gamma(a_\ell + \alpha_\ell m)}{\prod_{\ell=1}^q \Gamma(b_\ell + \beta_\ell m)} \frac{x^m}{m!}$$

for suitably bounded values of $|x|$, where the parameters involved satisfy

$$\alpha_\ell \in \mathbb{R}_+, \ell = \overline{1, p}; \quad \beta_j \in \mathbb{R}_+, j = \overline{1, q}; \quad \Delta_0 = 1 + \sum_{\ell=1}^q \beta_\ell - \sum_{j=1}^p \alpha_j > 0.$$

We also need the Srivastava-Daoust generalization of the Kampé de Fériet hypergeometric function in two variables defined by [15, p. 199]

$$\begin{aligned} & \mathcal{S}_{C:D:D'}^{A:B:B'} \left(\begin{matrix} [(a) : \theta, \Phi] : [(b) : \psi] ; [(b') : \psi'] \\ [(c) : \mu, \epsilon] : [(d) : \eta] ; [(d') : \eta'] \end{matrix} \middle| x, y \right) = \mathcal{S}_{C:D:D'}^{A:B:B'}(x, y) \\ & := \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^A \Gamma(a_j + m\theta_j + n\Phi_j) \prod_{j=1}^B \Gamma(b_j + m\psi_j) \prod_{j=1}^{B'} \Gamma(b'_j + n\psi'_j)}{\prod_{j=1}^C \Gamma(c_j + m\mu_j + n\epsilon_j) \prod_{j=1}^D \Gamma(d_j + m\eta_j) \prod_{j=1}^{D'} \Gamma(d'_j + n\eta'_j)} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.2) \end{aligned}$$

where the parameters satisfy

$$\theta_1, \dots, \theta_A, \dots, \eta'_1, \dots, \eta'_{D'} > 0.$$

For convenience, we write (a) to denote the sequence of A parameters $a_1, \dots, a_A$, with similar interpretations for $(b), \dots, (d')$. Empty products should be interpreted as unity. Srivastava and Daoust [16, p. 155] reported that the series in (1.2) converges absolutely for all $x, y \in \mathbb{C}$ when

$$\Delta_1 = 1 + \sum_{j=1}^C \mu_j + \sum_{j=1}^D \eta_j - \sum_{j=1}^A \theta_j - \sum_{j=1}^B \psi_j > 0,$$

$$\Delta_2 = 1 + \sum_{j=1}^C \epsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^A \Phi_j - \sum_{j=1}^{B'} \psi'_j > 0.$$

The complete set of conditions for convergence of the series $S_{C:D;D'}^{A:B;B'}(x, y)$ is given in Srivastava and Daoust [16, pp. 154-155]. Finally, we also need the logconvexity property of the gamma function (a direct consequence of the Hölder inequality):

$$\Gamma(\lambda x + (1 - \lambda)y) \leq \Gamma(x)^\lambda \Gamma(y)^{1-\lambda}, \quad \min(x, y) > 0, \lambda \in [0, 1]. \quad (1.3)$$

This inequality is sharp.

2. Main results

Theorem 1 derives an explicit closed form for the PDF of $\xi \sim S_\alpha(\beta, \gamma, \mu)$. Some corresponding asymptotic expansions are given in Theorem 2.

Theorem 1. *Let $\xi \sim S_\alpha(\beta, \gamma, \mu)$. Then for all $\alpha \in (1, 2]$ and $|\beta \tan(\pi \frac{\alpha}{2})| < 1 - \alpha^{-1}$ the PDF of ξ can be expressed as*

$$f(x) = \frac{1}{\pi \alpha \gamma} \Re \left\{ S_{0:0;0}^{1:0;0} \left(\begin{matrix} [1/\alpha : 1/\alpha; 1] : -; - \\ - : -; - \end{matrix} \middle| \frac{i(\mu - x)}{\gamma}, i\beta \tan(\pi \frac{\alpha}{2}) \right) \right\}. \quad (2.1)$$

Proof. Consider

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp \{i(\mu - x)t - \gamma^\alpha |t|^\alpha [1 - i\beta \Phi \operatorname{sgn}(t)]\} dt,$$

where $\Phi = \tan(\pi \frac{\alpha}{2})$, $i(\mu - x) =: X$ and $i\beta \gamma^\alpha \Phi =: Y$. The Maclaurin series of the two unimodular terms in the integrand gives us

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \left\{ \int_{\mathbb{R}} t^m (\operatorname{sgn}(t)|t|^\alpha)^n \exp \{-\gamma^\alpha |t|^\alpha\} dt \right\} \frac{X^m}{m!} \cdot \frac{Y^n}{n!} \\ &= \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \left\{ \int_{-\infty}^0 t^m (-(-t)^\alpha)^n \exp \{-\gamma^\alpha (-t)^\alpha\} dt \right. \\ &\quad \left. + \int_0^{\infty} t^{m+\alpha n} \exp \{-\gamma^\alpha t^\alpha\} dt \right\} \frac{X^m}{m!} \cdot \frac{Y^n}{n!} \\ &= \frac{1}{2\pi} \sum_{m,n=0}^{\infty} [1 + (-1)^{m+n}] \left\{ \int_0^{\infty} t^{m+\alpha n} e^{-(\gamma t)^\alpha} dt \right\} \frac{X^m}{m!} \cdot \frac{Y^n}{n!} \\ &= \frac{1}{2\pi \alpha \gamma} \sum_{m,n=0}^{\infty} [1 + (-1)^{m+n}] \left\{ \int_0^{\infty} t^{1/\alpha + m/\alpha + n-1} e^{-t} dt \right\} \\ &\quad \cdot \frac{(X\gamma^{-1})^m}{m!} \cdot \frac{(Y\gamma^{-\alpha})^n}{n!} \\ &= \frac{1}{2\pi \alpha \gamma} \sum_{m,n=0}^{\infty} [1 + (-1)^{m+n}] \Gamma\left(\frac{1+m}{\alpha} + n\right) \frac{(X\gamma^{-1})^m}{m!} \cdot \frac{(Y\gamma^{-\alpha})^n}{n!} \quad (2.2) \end{aligned}$$

This is equivalent to (2.1). Since $\Delta_1 = 1 - 1/\alpha$ and $\Delta_2 = 0$, a case not covered by Srivastava and Daoust [16], the series convergence needs further study. Substituting

$$\lambda := \frac{1}{\alpha}, \quad x := \frac{1}{2} + m, \quad y := \frac{\alpha}{\alpha - 1} \left(\frac{1}{2\alpha} + n \right)$$

into (1.3), we obtain the estimate

$$\Gamma \left(\frac{1}{\alpha} + \frac{1}{\alpha} m + n \right) \leq \left[\Gamma \left(\frac{1}{2} + m \right) \right]^{\frac{1}{\alpha}} \left[\Gamma \left(\frac{\frac{1}{2} + n\alpha}{\alpha - 1} \right) \right]^{1 - \frac{1}{\alpha}} \quad (2.3)$$

for all $m, n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Applying (2.3) to (2.2), we deduce

$$|f(x)| \leq \frac{1}{\pi\alpha\gamma} \sum_{m=0}^{\infty} \left[\Gamma \left(\frac{1}{2} + m \right) \right]^{\frac{1}{\alpha}} \frac{(|X|\gamma^{-1})^m}{m!} \cdot \sum_{n=0}^{\infty} \left[\Gamma \left(\frac{\frac{1}{2} + n\alpha}{\alpha - 1} \right) \right]^{1 - \frac{1}{\alpha}} \frac{(|Y|\gamma^{-\alpha})^n}{n!}.$$

The resulting power series' radii or convergence are $\rho_X = \infty$ and $\rho_Y = 1 - \alpha^{-1}$ respectively. Therefore, the first one converges for all $x \in \mathbb{R}$, while the second one converges for $|Y|\gamma^{-\alpha} = |\beta\Phi| < 1 - \alpha^{-1}$.

The proof is complete. $\square$

Theorem 2. Assume $\alpha \in (1, 2]$ and $|\beta \tan(\pi \frac{\alpha}{2})| \rightarrow 0$. Then we have

$$f(x) = \frac{1}{\sqrt{\pi\alpha\gamma}} {}_1\Psi_1 \left[\begin{matrix} (1/\alpha, 1/\alpha) \\ (1/2, 1) \end{matrix} \middle| - \left(\frac{x - \mu}{4\gamma} \right)^2 \right]. \quad (2.4)$$

Moreover,

$$f(\mu) = \frac{\Gamma \left(1 + \frac{1}{\alpha} \right)}{\pi\gamma} {}_2F_1 \left[\begin{matrix} \frac{1}{2\alpha}, \frac{1}{2} \left(\frac{1}{\alpha} + 1 \right) \\ \frac{1}{2} \end{matrix} \middle| - (\beta \tan(\pi \frac{\alpha}{2}))^2 \right] \quad (2.5)$$

for all $\alpha \in (1, 2]$ and $|\beta \tan(\pi \frac{\alpha}{2})| < 1 - \alpha^{-1}$.

Proof. Note that

$$\lim_{y \rightarrow 0} \mathcal{S}_{C:D;D'}^{A:B;B'}(x, y) = {}_{A+B}\Psi_{C+D} \left[\begin{matrix} ((a), \theta), ((b), \psi) \\ ((c), \mu), ((d), \eta) \end{matrix} \middle| x \right].$$

Now, letting $|\beta \tan(\pi \frac{\alpha}{2})| \rightarrow 0$ in (2.1), we see that

$$\begin{aligned} f(x) &= \frac{1}{\pi\alpha\gamma} \Re \left\{ \sum_{m=0}^{\infty} \Gamma \left(\frac{1+m}{\alpha} \right) \frac{[i(\mu - x)/\gamma]^m}{m!} \right\} \\ &= \frac{1}{\pi\alpha\gamma} \sum_{m=0}^{\infty} \Gamma \left(\frac{1+2m}{\alpha} \right) \frac{[-(x - \mu)^2/\gamma^2]^m}{(2m)!} \\ &= \frac{1}{\pi^{3/2}\alpha\gamma} \sum_{m=0}^{\infty} \frac{\Gamma \left(\frac{1}{\alpha} + \frac{2}{\alpha} m \right)}{\Gamma \left(\frac{1}{2} + m \right)} \frac{[-(x - \mu)/(4\gamma^2)]^m}{m!}, \end{aligned}$$

where we have applied $(2m)! = 2m\Gamma(2m) = 4^m\Gamma(m+1)\Gamma(m+\frac{1}{2})/\sqrt{\pi}$. So, we have (2.4) since the confluent Fox-Wright series, ${}_1\Psi_1$, converges for all $2\alpha > 1$.

Now, letting $x \rightarrow \mu$ in (2.1) and calculating the real parts of the resulting special functions, we arrive at

$$\begin{aligned} f(\mu) &= \frac{1}{\pi\alpha\gamma} \Re \left\{ \sum_{n=0}^{\infty} \Gamma\left(\frac{1}{\alpha} + n\right) \frac{[i\beta\Phi]^n}{n!} \right\} \\ &= \frac{1}{\pi\alpha\gamma} \sum_{n=0}^{\infty} \Gamma\left(\frac{1}{\alpha} + 2n\right) \frac{[-(\beta\Phi)^2]^n}{(2n)!} \\ &= \frac{2^{1/\alpha-1}}{\pi\alpha\gamma} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2\alpha} + n\right) \Gamma\left(\frac{1}{2\alpha} + \frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2} + n\right)} \frac{[-(\beta\Phi)^2]^n}{n!}, \end{aligned}$$

where we have applied the Legendre duplication formula, $\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})/\sqrt{\pi}$. To reduce the sum to a Gauss hypergeometric function, ${}_2F_1$, we need the Pochhammer symbols $(z)_n = \Gamma(z+n)/\Gamma(z)$:

$$f(\mu) = \frac{2^{1/\alpha-1}}{\pi^{3/2}\alpha\gamma} \Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\frac{1}{2\alpha} + \frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2\alpha}\right)_n \left(\frac{1}{2\alpha} + \frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n} \frac{[-(\beta\Phi)^2]^n}{n!}.$$

But, this is equivalent to (2.5) since ${}_2F_1$ converges inside the open unit disc – note from Theorem 1 that $|\beta\Phi| < 1 - \alpha^{-1}$. $\square$

Zolotarev [20, p. 87, §2.4] reported on the following inequality for the k th derivative of the PDF of $\mathcal{S}_\alpha(\beta, 1, 0)$:

$$\left| f^{(k)}(x) \right| \leq \frac{\Gamma\left(\frac{1+k}{\alpha}\right)}{\pi\alpha} \left(\cos\left[\frac{\pi}{2}\beta K(\alpha)\right] \right)^{-(1+k)/\alpha}, \quad k \in \mathbb{N} \quad (2.6)$$

for some constant $K(\alpha)$. Zolotarev did not discuss the background or the sharpness of (2.6). The following theorem generalizes this inequality giving a functional upper bound for the modulus of the derivatives of $f(x)$.

Theorem 3. *Let $\xi \sim \mathcal{S}_\alpha(\beta, \gamma, \mu)$. Then, for $\alpha \in (1, 2]$, $|\beta \tan(\pi\frac{\alpha}{2})| < 1 - \alpha^{-1}$ and for any $\kappa \in \mathbb{N}$, we have*

$$\left| f^{(\kappa)}(x) \right| \leq \frac{1}{\pi\alpha\gamma^{\kappa+1}} \mathcal{S}_{0:0;0}^{1:0;0} \left(\begin{matrix} [\frac{1+\kappa}{\alpha} : \frac{1}{\alpha}; 1] : -; - \\ -; -; - \end{matrix} \middle| \frac{|x-\mu|}{\gamma}, |\beta \tan(\pi\frac{\alpha}{2})| \right). \quad (2.7)$$

Proof. Following the proof of Theorem 1, we obtain:

$$f^{(\kappa)}(x) = \frac{(-i)^\kappa}{2\pi} \sum_{m,n=0}^{\infty} \left\{ \int_0^\infty t^{\kappa+m+\alpha n} e^{-\gamma^\alpha t^\alpha} dt \right\} [1 + (-1)^{\kappa+m+n}] \frac{X^m}{m!} \cdot \frac{Y^n}{n!}.$$

Thus,

$$\left| f^{(\kappa)}(x) \right| \leq \frac{1}{\pi\alpha\gamma^{\kappa+1}} \sum_{m,n=0}^{\infty} \Gamma\left(\frac{1+\kappa}{\alpha} + \frac{m}{\alpha} + n\right) \frac{[|x-\mu|\gamma^{-1}]^m}{m!} \cdot \frac{[|\beta\Phi|]^n}{n!}, \quad (2.8)$$

and (2.7) is proved, because the double series converges for the stated range of parameters. $\square$

Remark 1. *Alternatively, separate the gamma-term with the aid of (1.3) into*

$$\Gamma\left(\frac{1+\kappa}{\alpha} + \frac{m}{\alpha} + n\right) \leq \left[\Gamma\left(\frac{1+\kappa}{2} + m\right)\right]^{\frac{1}{\alpha}} \cdot \left[\Gamma\left(\frac{\frac{1+\kappa}{2} + \alpha n}{\alpha - 1}\right)\right]^{1 - \frac{1}{\alpha}}$$

by choosing

$$\lambda := \frac{1}{\alpha}, \quad x := \frac{1+\kappa}{2} + m, \quad y := \frac{\alpha}{\alpha-1} \left(\frac{1+\kappa}{2\alpha} + n\right).$$

Applying this estimate to the right hand double sum in (2.8), we conclude

$$\begin{aligned} |f^{(\kappa)}(x)| &\leq \frac{1}{\pi\alpha\gamma^{\kappa+1}} \sum_{m=0}^{\infty} \left[\Gamma\left(\frac{1+\kappa}{2} + m\right)\right]^{\frac{1}{\alpha}} \frac{[|x - \mu|\gamma^{-1}]^m}{m!} \\ &\quad \cdot \sum_{n=0}^{\infty} \left[\Gamma\left(\frac{\frac{1+\kappa}{2} + \alpha n}{\alpha - 1}\right)\right]^{1 - \frac{1}{\alpha}} \frac{[|\beta\Phi|]^n}{n!}. \end{aligned} \quad (2.9)$$

The involved power series' have the same radii of convergence for all $\kappa \in \mathbb{N}_0$. This is not the case for the ones in the proof of Theorem 1.

Unfortunately, upper bounds like $[\Gamma(x)]^\lambda \leq \Gamma(x)$, $\lambda \in [0, 1]$, $x \geq 2$ can not be used to estimate (2.9) since the resulting confluent hypergeometric function, ${}_1F_0$, is defined only in the interval $x \in (\mu - \gamma, \mu + \gamma)$ and the resulting Fox-Wright series, ${}_1\Psi_0$, is formal, converges only at $\beta\Phi = 0$.

3. Discussion and final remarks

The following asymptotic expansion for the PDF of $\mathcal{S}_\alpha(0, 1, 0)$

$$f(x) = \frac{1}{\pi\alpha} \sum_{m=0}^{\infty} (-1)^m \Gamma\left(\frac{1+2m}{\alpha}\right) \frac{(-x^2)^m}{(2m)!} \sim \frac{\Gamma(\frac{1}{\alpha})}{\pi\alpha}$$

for $x \rightarrow 0$ has been obtained by Bergström [1] and mentioned in Zolotarev [20, p. 94, Theorem 2.5.1, Corollary 2]. An exhaustive account of the asymptotics of the PDF is given in [20, Sections 2.4 and 2.5]. See also [18, Eq. (4.1.3)].

Our (2.5) in Theorem 2 generalizes these asymptotics of the PDF $f(x)$ for the nonstandard $\mathcal{S}_\alpha(\beta, \gamma, \mu)$ case when $|\beta \tan(\pi \frac{\alpha}{2})| < 1 - \alpha^{-1}$. When $|\beta \tan(\pi \frac{\alpha}{2})| \geq 1 - \alpha^{-1}$, we can use the analytical continuation principle. For this, consult the series expansion by Uchaikin and Zolotarev [18, p. 108, Eq. (4.2.10)].

Hoffmann-Jørgensen [4, Theorem, Eq. (2.2)] has also expressed the PDF of $\xi \sim \mathcal{S}_\alpha(\beta, \gamma, \mu)$ in terms of known special functions. Using the Mellin-transform technique and introducing the *generalized incomplete hypergeometric function*, ${}_pG_q$, Hoffmann-Jørgensen [4] obtained expressions for (2.4) and (2.5) when $\alpha \in (1, 2]$. For this method, consult [18, pp. 179-180, §6.7.].

Let us mention that Hoffmann-Jørgensen's function ${}_pG_q$ [4, Eq. (6)] is, in fact, Fox-Wright generalized ${}_p\Psi_q$ -function with equal parameters $0 < \gamma < 1$, that is

$$\begin{aligned} {}_pG_q(a_1, \dots, a_p; b_1, \dots, b_q; \gamma, z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \frac{\prod_{j=1}^p \frac{\Gamma(a_j + \gamma m)}{\Gamma(a_j)}}{\prod_{k=1}^q \frac{\Gamma(b_k + \gamma m)}{\Gamma(b_k)}} \\ &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} {}_p\Psi_q \left[\begin{matrix} (a_p, \gamma) \\ (b_q, \gamma) \end{matrix} \middle| z \right]. \end{aligned}$$

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