Integral stochastic orderings and associated random variables

Let X and Y be real random variables. Recall that the classical stochastic ordering $\leq^{st}$ goes as follows: $X \leq^{st} Y \;\Leftrightarrow\; P(X \geq t) \leq P(Y \geq t)\ \forall\, t \in \mathbf{R}$. If $\mu_X(B) = P(X \in B)$ and $\mu_Y = P(Y \in B)$ denote the distributions of X and Y , we have $X \leq^{st} Y$ if and only if $\int^* \phi\, d\mu_X \leq \int^* \phi\, d\mu_Y$ for all increasing functions $\phi : \mathbf{R} \to \mathbf{R}$. This suggests the following extension:

Let $\mathrm{Pr}(S, \mathcal{B})$ denote the set of all probability measures on the measurable space $(S, \mathcal{B})$ be a measurable space and let Φ be a given set of real-valued functions on S . If $\mu, \nu \in \mathrm{Pr}(S, \mathcal{B})$, we write $\mu \leq^{\Phi} \nu$ if and only if $\int^* \phi\, d\mu \leq \int^* \phi\, d\nu$. Then $\leq^{\Phi}$ is a preordering on $\mathrm{Pr}(S, \mathcal{B})$ which has proved to be a useful tool in many different problems. Some typical examples are (i): $S = \mathbf{R}^k$ and $\Phi =$ all increasing functions; (ii): S is an interval on $\mathbf{R}$ and $\Phi =$ all (increasing) convex functions; (iii) $S = \mathbf{R}^k$ and $\Phi =$ all (increasing) supermodular functions (recall that $\phi : \mathbf{R}^k \to \mathbf{R}$ is *supermodular* if $\phi(x) + \phi(y) \leq \phi(x \wedge y) + \phi(x \vee y)$ for all $x, y \in \mathbf{R}^k$).

Stochastic orderings are intimately connected to the concepts of positive and negative association of random variables. Let $X_I = (X_i \mid i \in I)$ be a family of real random variables. If $\alpha \subseteq I$ is finite we let $X_\alpha = (X_i)_{i \in \alpha}$ denote the marginal of X_I and we say that X_I is *positively (negatively) associated* if the covariance $\mathrm{Cov}(f(X_\alpha), h(X_\beta))$ is non-negative (non-positive) for all disjoint finite sets $\alpha, \beta \subseteq I$ and all increasing Borel functions $f : \mathbf{R}^\alpha \to \mathbf{R}$ and $h : \mathbf{R}^\beta \to \mathbf{R}$ for which the covariance exists. Note that $(X_i \mid i \in I)$ are independent if and only if X_I is positively and negatively associated. If $(X_i \mid i \in I)$ is a Gaussian process with covariance function $\sigma(i, j)$, it can be shown that X_I is positively associated if and only if $\sigma(i, j) \geq 0$ for all i, j and that X_I is negatively associated if and only if $\sigma(i, j) \leq 0$ for all $i \neq j$.

Positively or negatively associated sequences shares many properties with independent random variables and positive or negative association for a stationary sequence is closely related (but not comparable) to mixing conditions. In the lecture, I shall demonstrate how positive or negative association may replace independence in the law of large numbers and the central limit theorem.