

The Appearance of k -th order polynomial eigenvalue problems when damping is modeled by fractional derivatives

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Definition

Fractional Differential Equations

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} D^\alpha \mathbf{u} + \mathbf{K} \mathbf{u} = \mathbf{0}, \quad \mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{n \times n}. \quad (1)$$

Riemann-Liouville definition

$$D^\alpha \mathbf{u}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\mathbf{u}(t-\tau)}{\tau^\alpha} d\tau, \quad 0 < \alpha < 1 \quad (2)$$

$$\alpha = \frac{p}{q}, \quad p, q \in \mathbb{N}_0^+ \quad (3)$$

Augmented State-Space Representation

Composition Rule [7]

$$\mathbf{A}\mathbf{z} = \mathbf{B}D^\alpha \mathbf{z}, \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{2qn \times 2qn}, \quad \alpha = \frac{1}{2} \quad (4)$$

Suarez and Shokooh et al. [9, 8, 1, 5]

$$\begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{I}_n & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_n & \mathbf{O} & \mathbf{O} \\ \mathbf{I}_n & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & -\tilde{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_n \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_n & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_n & \mathbf{O} & \mathbf{O} \\ \mathbf{I}_n & \mathbf{O} & \mathbf{O} & \tilde{\mathbf{C}} \end{bmatrix} D^\alpha \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} \\ \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{M} \\ \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} \\ \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{C} \end{bmatrix} D^\alpha \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} \quad (6)$$

$$\mathbf{z}_4 = \mathbf{u}, \quad D^\alpha \mathbf{z}_1 = \ddot{\mathbf{u}}$$

Augmented State-Space Representation

$$\mathbf{A}z = \mathbf{B}D^\alpha z, \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{2qn \times 2qn}, \quad \alpha = \frac{2}{3} \quad (7)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{M} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{C} \\ \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{C} & \mathbf{O} \end{bmatrix}, \quad (8)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{C} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{K} \end{bmatrix} \quad (9)$$

Bottleneck and a Loophole

- The system of equations tends to become very large when the denominator q of the order $\alpha = \frac{p}{q}$ of the derivative is very large.
- Applying the Fourier transform to the system of equations (1), one obtains

$$\left[\mathbf{M} (i\omega)^2 + \mathbf{C} (i\omega)^\alpha + \mathbf{K} \right] \mathbf{x} = \mathbf{0}. \quad (10)$$

- It is convenient to denote the eigenvalues by $\lambda_j = (i\omega_j)^{\frac{1}{q}}$

$$\left[\mathbf{M} \lambda^{2q} + \mathbf{C} \lambda^p + \mathbf{K} \right] \mathbf{x} = \mathbf{0}. \quad (11)$$

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Extension

Fenander (1996) [3]

$$I M D^{2+\alpha} \mathbf{u} + M \ddot{\mathbf{u}} + C D^{\alpha} \mathbf{u} + K \mathbf{u} = \mathbf{f} + I D^{\alpha} \mathbf{f}. \quad (12)$$

Rational eigenvalue problem Voss (2006) [6]

$$\left(M \lambda^{2q} + \frac{C \lambda^p + K}{1 + I \lambda^p} \right) \mathbf{x} = \mathbf{0}. \quad (13)$$

Polynomial eigenvalue problem

$$\left(I M \lambda^{2q+p} + \lambda^{2q} M + \lambda^p C + K \right) \mathbf{x} = \mathbf{0}. \quad (14)$$

Chu's Homotopy

Polynomial eigenvalue problem

$$\mathbf{P}(\lambda) \mathbf{x} = \mathbf{0} \quad (15)$$

$$\mathbf{P}(\lambda) = \mathbf{A}_k \lambda^k + \mathbf{A}_{k-1} \lambda^{k-1} + \cdots + \mathbf{A}_1 \lambda + \mathbf{A}_0 \quad (16)$$

Nonlinear system $\mathbf{h}(\mathbf{x}, \lambda, t) = \mathbf{0}$ Chu (1988) [2]

$$\mathbf{h}(\mathbf{x}, \lambda, t) = \begin{bmatrix} \mathbf{R}(\lambda, t, \mathbf{D}) \mathbf{x} \\ \frac{1}{2}(\mathbf{x}^H \mathbf{x} - 1) \end{bmatrix} \quad (17)$$

$$\begin{aligned} \mathbf{R}(\lambda, t, \mathbf{D}, c) &= (1-t) \mathbf{Q}(\lambda) + t \mathbf{P}(\lambda), \\ \mathbf{Q}(\lambda) &= c \mathbf{I}_n \lambda^k - \mathbf{D}, \\ \mathbf{D} &= \text{diag}(d_j), \quad c \in \mathbb{C}, \quad d_j \in \mathbb{C}. \end{aligned} \quad (18)$$

Chu's Homotopy continued

$$\begin{bmatrix} \mathbf{R}(\lambda, t, \mathbf{D}, c) & \mathbf{R}_\lambda(\lambda, t, c) \mathbf{x} \\ \mathbf{x}^H & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} (\mathbf{Q}(\lambda) - \mathbf{P}(\lambda)) \mathbf{x} \\ 0 \end{bmatrix} \quad (19)$$

k n initial conditions $\mathbf{z} = \begin{bmatrix} \mathbf{x}_i \\ \lambda_{ij} \end{bmatrix}$ with

$$\mathbf{x}_i(t=0) = \mathbf{e}_i, \quad (20)$$

$$\lambda_{ij}(t=0) = \lambda_{ij} \quad i = 1, \dots, n, \quad j = 1, \dots, k. \quad (21)$$

The *initial* eigenvalues λ_{ij} for $t = 0$ follow from

$$\det \mathbf{Q}(\lambda) = \prod_{i=1}^n (c - \lambda^k d_i) = 0. \quad (22)$$

Chu's Homotopy continued

$$\begin{bmatrix} \mathbf{R}(\lambda, t, \mathbf{D}, c) & \mathbf{R}_\lambda(\lambda, t, c) \mathbf{x} \\ \mathbf{x}^H & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{R}(\lambda, t, \mathbf{D}, c) \mathbf{x} \\ \frac{1}{2} (\mathbf{x}^H \mathbf{x} - 1) \end{bmatrix}. \quad (23)$$

$$\begin{aligned} \mathbf{P}_\lambda &= k \lambda^{k-1} \mathbf{A}_k + (k-1) \lambda^{k-2} \mathbf{A}_{k-1} + \cdots + 2 \lambda \mathbf{A}_2 + \mathbf{A}_1, \\ \mathbf{Q}_\lambda &= c k \lambda^{k-1} \mathbf{I}_n, \end{aligned} \quad (24)$$

$$\mathbf{R}_\lambda(\lambda, t, c) = (1-t) \mathbf{Q}_\lambda(\lambda) + t \mathbf{P}_\lambda(\lambda).$$

- The coefficient matrix in Eq. (19,23) is of order $n + 1$.
- The parameters c, \mathbf{D} in Eq. (18) are random numbers.
- The homotopy curves correspond only to different initial values of the same ODE.
- Hence, all curves can be followed **simultaneously**.

Chu's Homotopy continued

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Example I

Agrawal (2002) [1]

$$m\ddot{u} + cD^{0.5}u + ku = 0 \quad (25)$$

$$\mathbf{A}\mathbf{X} = \mathbf{B}\mathbf{X}\mathbf{\Lambda}, \quad \mathbf{X}^T\mathbf{B}\mathbf{X} = \mathbf{I}_4, \quad \mathbf{\Lambda} = \text{diag}(\lambda_j), \quad (26)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k/m \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & c/m \end{bmatrix} \quad (27)$$

$$u(t) = \frac{1}{\sqrt{\pi t}} \sum_{j=1}^4 \mathbf{e}_4^T \mathbf{x}_j c_j + \sum_{j=1}^4 \lambda_j \mathbf{e}_4^T \mathbf{x}_j g_j(t) c_j, \quad (28)$$

$$\dot{u}(t) = \frac{1}{\sqrt{\pi t}} \sum_{j=1}^4 \mathbf{e}_2^T \mathbf{x}_j c_j + \sum_{j=1}^4 \lambda_j \mathbf{e}_2^T \mathbf{x}_j g_j(t) c_j. \quad (29)$$

Example I continued

Since $u(t)$ and $\dot{u}(t)$ remain bounded as $t \rightarrow 0$, we have **constraints** for c_j . Finally we obtain a closed form solution for the position and velocity in terms of eigenvalues and eigenvectors [9].

$$u(t) = \sum_{j=1}^4 \lambda_j (\mathbf{e}_4^T \mathbf{x}_j)^2 (\dot{u}_0 + \lambda_j^2 u_0) g_j(t), \quad (30)$$

$$\dot{u}(t) = \sum_{j=1}^4 \lambda_j^3 (\mathbf{e}_4^T \mathbf{x}_j)^2 (\dot{u}_0 + \lambda_j^2 u_0) g_j(t). \quad (31)$$

$$g_j(t) = \exp(\lambda_j^2 t) \left(1 + \operatorname{erf}(\lambda_j \sqrt{t}) \right) \quad (32)$$



Example I continued

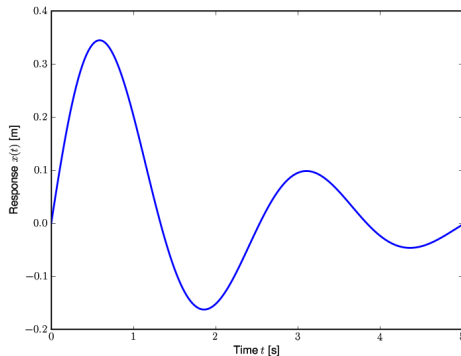


Figure: Free response of a fractionally damped oscillator,
 $u_0 = 0$, $\dot{u}_0 = 1.0$ m/s

Example II

Fenander (1996) [3]

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix},$$
$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}.$$

$$m_1 = 1\text{kg}, \quad m_2 = 2\text{kg}, \quad , k_1 = k_3 = 1\text{N/m}, \quad k_2 = 1.5\text{N/m},$$
$$c_1 = c_2 = c_3 = 0.4\text{Ns}^{2/3}/\text{m},$$

- $\alpha = \frac{2}{3}, \quad l = 0 \quad \text{Eq. (11)}$
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Numerical Results

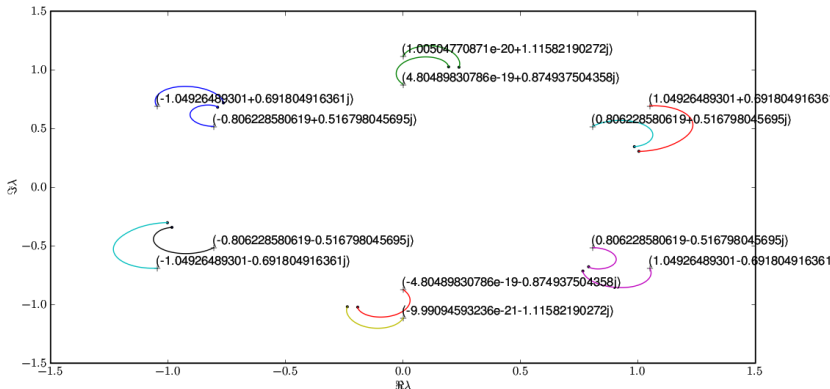






Figure: Homotopy curves for $\alpha = \frac{2}{3}$, $l = 0$

Conclusions and Outlook

- A direct approach based on a homotopy as suggested by Chu is applied to solve the polynomial eigenvalue problem.
- A predictor corrector method is used to solve the nonlinear problem (17). 
- ZVODE¹ might be another option to solve the complex initial value problem (19).
- Direct time-domain solutions are very time-consuming due to the non-local character of fractional damping [10]. 
- All computation were carried out by SciPy [4].



¹<http://www.llnl.gov/CASC/software.html> (see VODE)

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Thank you for your attention



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Software

<http://numpy.scipy.org/>

<http://www.scipy.org/>

<http://matplotlib.sourceforge.net/>